First, we have a probabilistic model $P(X, Z|\theta)$, where X is observed data, Z is unobserved data, θ is the parameter we would like to learn from data.

We want to get the maximum likelihood estimate for θ, this should be done by maximizing the likelihood $P(X, Z|\theta)$, unfortunately we don't know values for Z, so this is impossible.

Then we find that we can calculate $P(X|\theta)$ by summing out Z, $P(X|\theta) = \sum_Z P(X, Z|\theta)$. So the problem becomes how to find a θ that maximizes $P(X|\theta)$.

We are going to maximize $P(X|\theta)$ iteratively. At every iteration, we begin with a θ_{old}. We hope we can find a θ_{new} so that $P(X|\theta_{new}) > P(X|\theta_{old})$.

To do this, we need to prove an important inequality, that for any θ, we have

$$\begin{aligned}
\log P(X|\theta) &= \log \sum_Z P(X, Z|\theta) \\
&= \log \sum_Z P(Z|X, \theta_{old}) \frac{P(X, Z|\theta)}{P(Z|X, \theta_{old})} \\
&\geq \sum_Z P(Z|X, \theta_{old}) \log \frac{P(X, Z|\theta)}{P(Z|X, \theta_{old})} \\
&= \mathbb{E}_{Z|X,\theta_{old}} \log \frac{P(X, Z|\theta)}{P(Z|X, \theta_{old})} \\
&= \mathbb{E}_{Z|X,\theta_{old}} \log P(X, Z|\theta) - \mathbb{E}_{Z|X,\theta_{old}} \log P(Z|X, \theta_{old}) = Q(\theta) \ .
\end{aligned}$$

So we have proved that for any θ, $\log P(X|\theta) \geq Q(\theta)$.

We can also prove that $Q(\theta_{old}) = \log P(X|\theta_{old})$.

$$\begin{aligned}
Q(\theta_{old}) &= \mathbb{E}_{Z|X,\theta_{old}} \log P(X, Z|\theta_{old}) - \mathbb{E}_{Z|X,\theta_{old}} \log P(Z|X, \theta_{old}) \\
&= \mathbb{E}_{Z|X,\theta_{old}} \log \frac{P(X, Z|\theta_{old})}{P(Z|X, \theta_{old})} \\
&= \mathbb{E}_{Z|X,\theta_{old}} \log \frac{P(X, Z, \theta_{old})/P(\theta_{old})}{P(X, Z, \theta_{old})/P(X, \theta_{old})} \\
&= \mathbb{E}_{Z|X,\theta_{old}} \log \frac{P(X, \theta_{old})}{P(\theta_{old})} = \mathbb{E}_{Z|X,\theta_{old}} \log P(X|\theta_{old}) = \log P(X|\theta_{old}) \ .
\end{aligned}$$

Then we define θ_{new} to be

$$\theta_{new} = \arg\max_\theta Q(\theta),$$

so we know that for any θ, $Q(\theta_{new}) \geq Q(\theta)$.

As a summary, we have proved that

- for any θ, $\log P(X|\theta) \geq Q(\theta)$
- for any θ, $Q(\theta_{new}) \geq Q(\theta)$, where $\theta_{new} = \arg\max_\theta Q(\theta)$
- $Q(\theta_{old}) = \log P(X|\theta_{old})$

Therefore, $\log P(X|\theta_{new}) \geq Q(\theta_{new}) \geq Q(\theta_{old}) = \log P(X|\theta_{old})$. Note that when $P(X|\theta_{new}) = P(X|\theta_{old})$ the algorithm stops, so in other cases, $P(X|\theta_{new}) > P(X|\theta_{old})$.

As a final remark, since $\mathbb{E}_{Z|X,\theta_{old}} \log P(Z|X, \theta_{old})$ in $Q(\theta)$ is a constant, in practice we define θ_{new} as

$$\theta_{new} = \arg\max_\theta \ \mathbb{E}_{Z|X,\theta_{old}} \log P(X, Z|\theta) \ .$$